Construction of Commuting Difference Operators for Multiplicity Free Spaces

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1. Introduction

The analysis of invariant differential operators on certain multiplicity free spaces led recently to the introduction of a family of symmetric polynomials that is more general than Jack polynomials (see [KS], but also [OO1], [OO2]). They are called *interpolation Jack polynomials*, *shifted Jack polynomials*, or *Capelli polynomials*. Apart from being inhomogeneous, they are distinguished from classical Jack polynomials by their very simple definition in terms of vanishing conditions.

One of the most important and non-obvious properties of Capelli polynomials is that they are eigenfunctions of certain explicitly given difference (as opposed to differential) operators (see [KS]). This readily implies that their top homogeneous term is in fact a (classical) Jack polynomial. Other consequences include a binomial theorem, a Pieri formula, and much more.

It is well-known that Jack polynomials are tied to root systems of type A and that they have natural analogues for other root systems (see e.g., [He]). Therefore, it is a natural problem as to whether this holds for the Capelli polynomials as well. Okounkov [Ok] proposed such an analogue for root systems of type BC and proved that these share some of the nice properties of Capelli polynomials. But, unfortunately, Okounkov's polynomials do not satisfy difference equations. Also their representation-theoretic significance is not clear.

In this paper we go back to the origin and let ourselves be guided by the theory of multiplicity free actions. It is known (see section 7 for details) that these actions give rise to combinatorial structures consisting of four data $(\Gamma, \Sigma, W, \ell)$. Here Γ is a lattice, $\Sigma \subset \Gamma$ a basis, $W \subseteq \operatorname{Aut}\Gamma$ a finite reflection group, and $\ell \in \Gamma$ some element. These data alone suffice to *formulate* the definition of a (generalized) Capelli polynomial but, in that generality, neither existence nor uniqueness will hold, let alone any other good property.

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It is known that for structures coming from multiplicity free actions, Capelli polynomials *are* well-defined. Moreover, two of the most important cases (the classical,citeSymCap and the semiclassical [Kn2], see tables in section 8) have previously been worked out in detail. In particular, it was shown that the corresponding Capelli polynomials are eigenfunctions of certain difference operators.

In this paper we handle all other multiplicity free actions in an axiomatic fashion. We extracted from the multiplicity free case nine properties **C1**–**C9** and use them as the foundation of the theory. The main result is the construction of a commuting family of difference operators that is diagonalized by the Capelli polynomials. In a forthcoming paper, we study the algebra of difference operators in more detail and derive an evaluation formula, an explicit interpolation formula, and a binomial formula, among others.

It should be mentioned that the actual verification of properties **C1**–**C9** requires some case-by-case analysis which uses the classification results of [Kac], Benson-Ratcliff [BR1], and Leahy [Le]. On the other side, this disadvantage is offset by two things: first, other structures $(\Gamma, \Sigma, W, \ell)$ which do not come from multiplicity free actions may and do satisfy the axioms. Thus the theory developed in this paper has applications beyond multiplicity free actions even though the exact scope is not yet known.

Secondly, as kind of a byproduct, all Capelli polynomials depend on at least one free parameter. The polynomials that are actually attached to a multiplicity free space correspond to one particular choice of the parameters. This extra generality does not seem possible when working directly with the multiplicity free action.

There is another paper, [BR2] by Benson and Ratcliff, which studies eigenvalues of invariant differential operators on multiplicity free spaces from a combinatorial point of view. It is just opposite in its approach: multiplicity free spaces are treated conceptually but there are no parameters. Moreover, only the special values $p_\mu(\rho + \lambda)$ are investigated and not their interpolation $p_\mu(z)$. Nevertheless, the influence of [BR2] on the present treatment is acknowledged. This holds in particular for formula (6), apparently due to Yan [Yan], and the realization of how much can be deduced from it.

Finally, one important difference from the Jack case should be mentioned. This paper does *not* achieve the goal to define a shifted version of generalized Jacobi polynomials in the sense of Heckman [He] (i.e., analogues of Jack polynomials for other root systems): in general, the top homogeneous components of Capelli polynomials are new. Nevertheless, these components share a lot of properties with Jacobi polynomials such as being eigenfunctions for certain commuting differential operators. A unifying concept would be very desirable.

2. Data and axioms*

The goal of this paper is to study special polynomials that are constructed from the following data:

- a lattice Γ of finite rank;
- a basis Σ of Γ;
- a finite group $W \subseteq \mathrm{Aut}(\Gamma)$;
- an element $\ell \in \Gamma$.

Let $V := \mathrm{Hom}(\Gamma, \mathbb{C})$ and let $\mathcal{P} = S^\bullet(\Gamma \otimes \mathbb{C})$ denote the polynomial functions on V. The dual lattice $\Gamma^\vee$ sits inside V. Let $\Sigma^\vee \subseteq \Gamma^\vee$ be the basis dual to Σ and $\Lambda_+ := \sum \mathbb{N}\Sigma^\vee$ the monoid generated by it.

The structure $(\Gamma, \Sigma, W, \ell)$ is subject to the following conditions:

C1 *The group W is generated by reflections on V.*

Thus, W gives rise to a unique root system $\Delta \subseteq \Gamma$ such that all roots are primitive vectors. Let $\Delta^\vee \subseteq \Gamma^\vee$ be the set of coroots. Let $\Delta^+ := \{\alpha \in \Delta \mid \alpha(\Sigma^\vee) \geq 0\}$.

C2 $\Delta = \Delta^+ \cup (-\Delta^+)$

In other words, Δ^+ is a system of positive roots and all elements of $\Sigma^\vee$ are dominant with respect to it. Let $\Phi := W\Sigma$ and $\Phi^+ := \{\omega \in \Phi \mid \omega(\Sigma^\vee) \geq 0\}$.

C3 $\Phi \subseteq \Phi^+ \cup (-\Phi^+)$.

C4 $\ell \in \Gamma^W$.

C5 $\sum \Phi^+ - \sum \Delta^+ = \ell$.

C6 $\ell(\eta) > 0$ *for all* $\eta \in \Sigma^\vee$.

Let $\Sigma_1^\vee := \{\gamma \in \Sigma^\vee \mid \ell(\gamma) = 1\}$.

C7 *If* $\eta \in \Sigma_1^\vee$ *and* $\omega \in \Delta \cup \Phi$ *then* $\omega(\eta) \in \{-1, 0, 1\}$.

C8 *Any linear W-invariant on V is uniquely determined by its restriction to $\Sigma_1^\vee$.*

Let $\pm W$ be the group generated by W and -1. Then we define

$$V_0 := \{\rho \in V \mid \text{For all } \omega_1, \omega_2 \in \Sigma \text{ with } \omega_1 \in \pm W\omega_2 \text{ holds } \omega_1(\rho) = \omega_2(\rho)\}$$

* At first glance, these data and axioms might not seem very natural. Therefore, the reader may wish to look first at sections 7 and 8 for motivational background and examples.

Thus, for $\rho \in V_0$ and for every $\omega \in \Phi \cup (-\Phi)$ we can define $k_\omega := \omega_1(\rho)$ where $\omega_1 \in \pm W\omega \cap \Sigma$. In particular we have $k_\omega = k_{-\omega}$ for all $\omega \in \Phi$. Recall that ρ is called *dominant* (resp. *regular*) if $\alpha(\rho) \notin \mathbb{Z}_{<0}$ (resp. $\alpha(\rho) \neq 0$) for all $\alpha \in \Delta^+$. The last axiom is:

C9 *There is a regular dominant $\rho \in V_0$ with the following property: for every $\lambda \in \Lambda_+$ there is a unique polynomial $p \in \mathcal{P}^W$ of degree $\ell(\lambda)$ such that $p(\rho + \mu) = \delta_{\lambda\mu}$ (Kronecker delta) for all $\mu \in \Lambda_+$ with $\ell(\mu) \leq \ell(\lambda)$.*

We will show (Theorem 3.6) that these polynomials then exist in fact for every dominant $\rho \in V_0$.

3. The difference Euler operator

For any $\eta \in \Lambda_1 := W\Sigma_1^\vee$ define the following rational function on V:

$$f_\eta(z) := \frac{\prod\limits_{\omega \in \Phi : \omega(\eta) > 0} (\omega(z) - k_\omega)}{\prod\limits_{\alpha \in \Delta : \alpha(\eta) > 0} \alpha(z)}.$$

For any $\eta \in \Gamma$ we have the shift operator T_η on $\mathcal{P}$ defined by $(T_\eta f)(z) = f(z - \eta)$. Then we can define the difference operator $L := \sum_{\eta \in \Lambda_1} f_\eta(z) T_\eta$.

3.1. Proposition. *The operator L preserves the space of W-invariant polynomials.*

Proof: Let $f \in \mathcal{P}^W$. From $\rho \in V_0$ it follows that $L(f)$ is a W-invariant rational function. By **C2**, the ideal of W-skew-invariants is generated by $\delta = \prod_{\alpha \in \Delta^+} \alpha$. It follows from the definition of L that $\delta L(f)$ is a skew-invariant polynomial. Thus $L(f)$ is a W-invariant polynomial. □

If ρ is regular dominant, then $\alpha(\rho + \lambda) \neq 0$ for all $\lambda \in \Lambda_+$. Thus, $f_\eta(\rho + \lambda)$ is defined. The main property of f_η is the following cut-off property:

3.2. Lemma. *Assume ρ is regular dominant. Let $\eta \in \Lambda_1$ and $\lambda \in \Lambda_+$ with $\mu := \lambda - \eta \notin \Lambda_+$. Then $f_\eta(\rho + \lambda) = 0$.*

Proof: Since $\mu \notin \Lambda_+$ there is $\omega \in \Sigma$ with $\omega(\mu) = \omega(\lambda) - \omega(\eta) < 0$. We have $\omega(\lambda) \geq 0$ because $\lambda \in \Lambda_+$. Thus, $\omega(\eta) > 0$ and therefore $\omega(\eta) = 1$, by **C7**. This implies $\omega(\lambda) = 0$. Therefore, the factor $\omega(z) - k_\omega$ of f_η vanishes in $z = \rho + \lambda$. □

This has the following consequence:

3.3. Corollary. *Assume ρ to be regular dominant. For every $\lambda \in \Lambda_+$ let M_λ be the space of functions $f \in \mathcal{P}^W$ with $f(\rho + \mu) = 0$ for every $\mu \in \Lambda^+$ with $\ell(\mu) \leq \ell(\lambda)$ and $\mu \neq \lambda$. Then $L(M_\lambda) \subseteq M_\lambda$.*

Now we define the *difference Euler operator* as $E := \ell - L$. Clearly, it inherits the properties expressed in the last two propositions from L. Additionally:

3.4. Proposition. *If* $f \in \mathcal{P}^W$ *then* $\deg E(f) \leq \deg f$.

Proof: Let $\eta \in \Sigma_1^\vee$. Then $\omega \in \Phi$, $\omega(\eta) > 0$ implies $\omega \in \Phi^+$ (from **C3**). Similarly for Δ (from **C2**). Since, by **C5**, we have $\sum_{\omega\in\Phi^+}\omega(\eta) = 1 + \sum_{\alpha\in\Delta^+}\alpha(\eta)$ we conclude (from **C7**) that there is one more $\omega \in \Phi$ with $\omega(\eta) > 0$ than $\alpha \in \Delta$ with $\alpha(\eta) > 0$. Thus f_η is a rational function of degree one. By W-equivariance, the same holds for all $\eta \in \Lambda_1$. This shows $\deg L(f) \leq \deg f + 1$.

We examine the effect of L on the highest degree component of f. There, T_η acts as identity. Hence, L acts as multiplication by a W-invariant linear function $\ell'(z)$ which is independent of ρ. It remains to be shown that $\ell' = \ell$.

Let $\eta_1 \in \Sigma_1^\vee$. We enumerate the other elements of $\Sigma^\vee$ as $\eta_2, \ldots, \eta_r$. Let $\omega_i \in \Sigma$ such that $\omega_i(\eta_j) = \delta_{ij}$. When we write $\ell = \sum_i a_i\omega_i$, then $a_i = \ell(\eta_i) > 0$ by **C6**. Substituting any $\eta \in \Lambda_1$ we obtain

$$1 = \ell(\eta) = \omega_1(\eta) + \sum_{i\geq 2} a_i\omega_i(\eta) \leq 1 + \sum_{i\geq 2} a_i\omega_i(\eta).$$

Assume $\omega_i(\eta) \leq 0$ for all $i \geq 2$. Then $\omega_i(\eta) = 0$ for all $i \geq 2$ which implies that η is a multiple of η_1. Because $\ell(\eta) = \ell(\eta_1) = 1$ we get $\eta = \eta_1$. Thus, for $\eta \neq \eta_1$ there is $i \geq 2$ with $\omega_i(\eta) > 0$. Therefore, the factor $\omega_i(z) - k_{\omega_i}$ appears in the definition of f_η, which implies that $f_\eta(\rho + \eta_1) = 0$.

Now we apply L to $f = 1$ and obtain

$$\sum_\eta f_\eta(z) = L(1) = \ell'(z) + a(\rho) \tag{1}$$

where $a(\rho)$ is some constant. For $\rho = 0$ the left-hand side of (1) becomes homogeneous, which implies $a(0) = 0$. But first we substitute $z = \rho + \eta_1$ in (1) where $\eta_1 \in \Sigma_1^\vee$ and get $f_{\eta_1}(\rho + \eta_1) = \ell'(\rho + \eta_1) + a(\rho)$. Now we put $\rho = 0$ and get $1 = \ell'(\eta_1)$ (by **C7**). Thus $\ell(\eta_1) = \ell'(\eta_1)$ for all $\eta_1 \in \Sigma_1^\vee$. Since both ℓ and ℓ' are W-invariant, **C8** implies $\ell = \ell'$. $\square$

3.5. Lemma. *The action of* E *on* $\mathcal{P}^W$ *is diagonalizable. Moreover, if* ρ *is dominant and* $g \in \mathcal{P}^W$ *is an eigenvector of* E*, then its eigenvalue equals*

$$\ell(\rho) + \min\{\ell(\lambda) \mid \lambda \in \Lambda_+, g(\rho + \lambda) \neq 0\}. \tag{2}$$

Proof: Assume first that ρ regular and dominant. For $d \in \mathbb{N}$ let U_d be the space of $g \in \mathcal{P}^W$ with $g(\rho + \mu) = 0$ for all $\mu \in \Lambda_+$ with $\ell(\mu) < d$. These spaces form a decreasing filtration of $\mathcal{P}^W$. We have

$$E(g)(z) = \ell(z)g(z) - \sum_\eta f_\eta(z)g(z - \eta).$$

Thus, Lemma 3.2 implies that each U_d is E-stable. Moreover, $E - \ell(\rho) - d$ maps U_d into U_{d+1}. This means that E acts on U_d/U_{d+1} as scalar multiplication by $\ell(\rho) + d$. In particular, $\ell(\rho) + d$ is not an eigenvalue of E in U_{d+1}. The action of E is locally finite, since it preserves the degree. This implies that there is a unique E-stable complement $\overline{U}_d$ of U_{d+1} in U_d. Clearly, $\overline{U}_d$ is the (generalized) eigenspace of E in $\mathcal{P}^W$ with eigenvalue $\ell(\rho) + d$. Since the intersection of all U_d's is 0, there are no other other eigenvalues. Thus $\mathcal{P}^W = \oplus_d \overline{U}_d$.

Now assume that ρ is not regular or dominant. For $N \in \mathbb{N}$, let $\mathcal{P}^W_N$ be the E-stable space of invariant polynomials of degree $\leq N$. For any d choose an N_d such that $\mathcal{P}^W_N + U_d = \mathcal{P}^W$ for all $N \geq N_d$.

If ρ is only dominant, then the map $\rho + \Lambda_+ \to V/W$ is injective. Thus, the codimension of U_d is independent of ρ. This implies that for any $N \geq N_d$, the intersection $U_d \cap \mathcal{P}^W_N$ forms a family of subspaces of the finite dimensional space $\mathcal{P}^W_{N_d}$ which depends continuously on ρ. Also E depends continuously on ρ. It follows that $U_d \cap \mathcal{P}^W_N$, hence U_d itself, is E-stable. Then we conclude as above.

Finally, for given N and generic ρ choose d such that $\mathcal{P}^W_N \cap U_d = 0$. Then $\mathcal{P}^W_N$ is killed by $p(E)$, where $p(z) := \prod_{i=0}^{d-1}(z - \ell(\rho) - i)$. Again by continuity, $\mathcal{P}^W_N$ is killed for all ρ. Since p has no multiple zeros, E is diagonalizable on $\mathcal{P}^W_N$, hence on $\mathcal{P}^W$. $\square$

So far, we did not use condition **C9**. Now, it will provide the link between the function ℓ and the degree of a polynomial.

3.6. Theorem. *Let $\rho \in V_0$ be dominant.*

- *a) For every $\lambda \in \Lambda_+$ there is a unique polynomial $p_\lambda \in \mathcal{P}^W$ with $\deg p_\lambda \leq \ell(\lambda)$ and $p_\lambda(\rho + \mu) = \delta_{\lambda\mu}$ (Kronecker delta) for all $\mu \in \Lambda_+$ with $\ell(\mu) \leq \ell(\lambda)$.*
- *b) For every $d \in \mathbb{N}$, the set of p_λ with $\ell(\lambda) \leq d$ forms a basis of the space of $p \in \mathcal{P}^W$ with $\deg p \leq d$.*
- *c) The polynomial p_λ is an eigenvector for E. More precisely, $E(p_\lambda) = \ell(\rho + \lambda)p_\lambda$.*

Proof: We show first that *a)* implies *b)* and *c)*.

Let $\sum_\lambda a_\lambda p_\lambda = 0$ be a non-trivial linear dependence relation. Choose λ with $a_\lambda \neq 0$ and $\ell(\lambda)$ maximal. Then evaluation at $\rho + \lambda$ yields the contradiction $a_\lambda = 0$. Thus, the p_λ's are linearly independent.

Next, let $g \in \mathcal{P}^W$ with $\deg g = d$. By induction we may assume that *b)* holds for $d - 1$. Hence there is a linear combination g' of p_μ's with $\ell(\mu) \leq d - 1$ such that $h := g - g'$ vanishes at all points $\rho + \mu$ with $\ell(\mu) < d$. Thus $h' := h - \sum_{\ell(\lambda)=d} h(\rho + \lambda)p_\lambda$ vanishes in all points $\rho + \mu$ with $\ell(\mu) \leq d$. We have $\deg h' \leq d$. Thus, $h' \neq 0$ contradicts the uniqueness of p_λ with $\ell(\lambda) = d$. Thus g is a linear combination of the p_λ with $\ell(\lambda) \leq d$ which shows *b)*.

For *c)* it suffices, by continuity, to consider ρ regular and dominant. Consider the space M_λ from Corollary 3.3. By *a)*, it contains, up to a scalar only one polynomial of degree less at most $\ell(\lambda)$, namely p_λ. Thus, Corollary 3.3 and Proposition 3.4 imply that p_λ is an eigenvector for E. A direct calculation shows $E(p_\lambda)(\rho+\lambda) = \ell(\rho+\lambda)$. Hence the eigenvalue is $\ell(\rho+\lambda)$, which shows *c)*.

Now we prove *a)*. Condition **C9** guarantees the existence of one regular dominant ρ for which *a)*, hence *b)* and *c)* hold. Let $\Lambda_+(d)$ be the set of $\mu \in \Lambda_+$ with $\ell(\mu) \le d$. Let $\mathcal{P}_d^W$ be the space of $g \in \mathcal{P}^W$ with $\deg g \le d$. Then *c)* shows in particular that $\dim \mathcal{P}_d^W = \#\Lambda_+(d)$. Thus p_λ is defined by as many (inhomogeneous) linear equations as there are variables. Its unique solvability can be expressed by the non-vanishing of a determinant. This implies that *a)* holds for ρ in the complement of countably many hypersurfaces of V_0. This is, in particular, a Zariski dense subset of V_0.

Now consider the action of E on $\mathcal{P}_d^W$. Then, by *c)*, $\prod_{i=0}^d (E - \ell(\rho) - i)$ is zero on $\mathcal{P}_d^W$ for a Zariski dense subset of ρ's, hence for all $\rho \in V_0$. Let F_i be kernel of $E - \ell(\rho) - i$ in $\mathcal{P}_d^W$. Then $\mathcal{P}_d^W = \oplus_{i=0}^d F_i$. The dimension of F_i depends upper semicontinuously on ρ. On the other hand their sum is constant. Hence $\dim F_i$ is constant and equals the number N_d of $\mu \in \Lambda_+$ with $\ell(\mu) = d$.

Since ρ is dominant, (2) implies that every $g \in F(d)$ vanishes in $\rho + \Lambda_+(d-1)$. Moreover, the map $F(d) \to \mathbb{C}^{N_d} : g \mapsto (g(\rho+\lambda) \mid \ell(\lambda) = d)$ is injective. Since both sides have the same dimension it is also surjective. This implies that polynomials p_λ as in *a)* exist. Uniqueness follows again from the fact that the number of equations equals the number of variables. □

We record this last fact for future reference:

3.7. Corollary. *Let $\rho \in V_0$ be dominant. For every d, the dimension of the space of $g \in \mathcal{P}^W$ with $\deg g \le d$ equals the number of $\mu \in \Lambda_+$ with $\ell(\mu) \le d$.*

The equality $E(p_\lambda) = \ell(\rho+\lambda) p_\lambda$ can be rewritten as

$$\ell(z-\rho-\lambda) p_\lambda(z) = \sum_{\eta \in \Lambda_1} f_\eta(z) p_\lambda(z-\eta). \tag{3}$$

From this we obtain a formula for special values of p_λ. We need:

Definition: A *path* from $\lambda \in \Gamma$ to $\mu \in \Gamma$ is a sequence $\tau_* = \tau_0, \tau_1, \ldots, \tau_d \in V$ with $\tau_0 = \lambda$, $\tau_d = \mu$ and $\tau_i - \tau_{i-1} \in \Lambda_1$ for all $i = 1, \ldots, d$. The path is *positive* if all τ_i are in Λ_+.

Definition: An element $\rho \in V_0$ is *non-integral* if $\alpha(\rho) \notin \mathbb{Z}$ for all $\alpha \in \Delta$.

Observe that every ρ coming from a multiplicity free space is regular dominant but none is non-integral. Thus, certain formulas below are actually easier for non-geometric ρ-shifts.

3.8. Theorem. *Let $\rho \in V_0$ be non-integral. Let $\lambda, \mu \in \Lambda_+$ with $d = \ell(\mu - \lambda) \geq 0$. Then*

$$p_\lambda(\rho + \mu) = \frac{1}{d!} \sum_{\tau_*} f_{\tau_1 - \tau_0}(\rho + \tau_1) f_{\tau_2 - \tau_1}(\rho + \tau_2) \ldots f_{\tau_d - \tau_{d-1}}(\rho + \tau_d), \tag{4}$$

where the sum runs over all paths from λ to μ. Moreover, only positive paths contribute to the sum. Thus, if one restricts the sum to positive paths, then the formula is valid for all regular dominant ρ.

Proof: The non-integrality of ρ makes sure that none of the denominators vanish. We proceed by induction on d. The statement holds by definition for $d = 0$. Let $d \geq 1$. Putting $z = \rho + \mu$ in (3) we obtain

$$p_\lambda(\rho + \mu) = \frac{1}{d} \sum_{\eta \in \Lambda_1} f_\eta(\rho + \mu) p_\lambda(\rho + \mu - \eta).$$

By Lemma 3.2, the coefficient $f_\eta(\rho + \mu)$ is zero whenever $\tau_{d-1} = \mu - \eta$ is not in Λ_+. We conclude by induction. $\square$

As a corollary we get the extra vanishing property:

3.9. Corollary. *Let $\rho \in V_0$ be dominant. Let Λ be the monoid generated by Λ_1. Then $p_\lambda(\rho + \mu) = 0$ for every $\lambda, \mu \in \Lambda_+$ with $\mu - \lambda \notin \Lambda$.*

For fixed $k \geq 0$ we can sum over all paths with $\tau_k = \tau$ first. Then we get

3.10. Corollary. *Assume $\rho \in V_0$ is regular dominant. Then for all $\lambda, \mu \in \Lambda_+$ and all $k \in \mathbb{N}$:*

$$\binom{\ell(\mu - \lambda)}{k} p_\lambda(\rho + \mu) = \sum_{\substack{\tau \in \Lambda_+ \\ \ell(\tau - \lambda) = k}} p_\lambda(\rho + \tau) p_\tau(\rho + \mu). \tag{5}$$

Observe that both sides of (5) depend polynomially on μ. Thus, if we set $\mu = z - \rho$, we obtain the following Pieri type formula:

3.11. Corollary.

$$\binom{\ell(z) - \ell(\rho + \lambda)}{k} p_\lambda(z) = \sum_{\substack{\tau \in \Lambda_+ \\ \ell(\tau - \lambda) = k}} p_\lambda(\rho + \tau) p_\tau(z). \tag{6}$$

For $\lambda = 0$ we get

3.12. Corollary.

$$\binom{\ell(z) - \ell(\rho)}{k} = \sum_{\tau \in \Lambda_+ : \ell(\tau) = k} p_\tau(z).$$

4. Pieri rules

We consider the matrix of multiplication by $h \in \mathcal{P}^W$ in the p_λ-basis:

$$h(z)p_\mu(z) = \sum_{\eta\in\Gamma} a^h_\eta(\mu)p_{\mu+\eta}(z) \tag{7}$$

where we put $a^h_\eta(\mu) = 0$ whenever $\mu + \eta \notin \Lambda_+$. We can compute the coefficients by evaluating both sides in the points $z \in \rho + \Lambda_+$. The next proposition shows in particular that the sum is over a finite set of η's which is independent of μ.

4.1. Proposition. *$a^h_\eta(\mu) = 0$ unless $\eta \in \Lambda$ and $\ell(\eta) \le \deg h$.*

Proof: Fix $\mu \in \Lambda_+$ and choose $\eta_0 \in \Gamma$ with $\ell(\eta_0)$ minimal such that $\eta_0 \notin \Lambda$ but $a^h_{\eta_0}(\mu) \neq 0$. In particular, $\mu + \eta_0 \in \lambda_+$. Substituting $z = \rho + \mu + \eta_0$ in (7) we obtain by the extra vanishing property (Corollary 3.9) that

$$0 = h(\rho + \mu + \eta_0)p_\mu(\rho + \mu + \eta_0) = a^h_{\eta_0}(\mu) + \sum_{\eta\neq\eta_0} a^h_\eta(\mu)p_{\mu+\eta}(\rho + \mu + \eta_0).$$

The p-factor in the sum vanishes unless $\eta_0 - \eta \in \Lambda$. This implies $\eta \notin \Lambda$ and $\ell(\eta) < \ell(\eta_0)$. From this we get $a^h_\eta(\mu) = 0$ by minimality. Thus $a^h_{\eta_0}(\mu) = 0$.

The inequality $\ell(\eta) \le \deg h$ simply reflects the fact that the p_μ with $\ell(\mu) \le d$ form a basis of the space of invariant polynomials of degree $\le d$ (Theorem 3.6b). $\square$

4.2. Theorem. *Let $\rho \in V_0$ be non-integral. Let $\tau \in \Lambda$ with $d := \ell(\tau)$ and $\mu \in \Lambda_+$ with $\mu + \tau \in \Lambda_+$. Then*

$$a^h_\tau(\mu) = \sum_{\tau_*} \left[\sum_{i=0}^{d} \frac{(-1)^{d-i}}{i!(d-i)!} h(\rho + \tau_i)\right] f_{\tau_1-\tau_0}(\rho + \tau_1)\dots f_{\tau_d-\tau_{d-1}}(\rho + \tau_d) \tag{8}$$

where the outer sum runs over all paths from μ to $\mu + \tau$. Moreover, only positive paths contribute to the sum. Thus, if one restricts the sum to positive paths, then the formula is valid for all regular dominant ρ.

Proof: Substituting $z = \rho + \mu$ in (7) yields $a^h_0(\mu) = h(\rho + \mu)$ which is just (8) for $\tau = 0$. Now we proceed by induction on d.

In (7), we substitute $z = \rho + \mu + \tau$ and obtain

$$h(\rho + \mu + \tau)p_\mu(\rho + \mu + \tau) = \sum_\eta a^h_\eta(\mu)p_{\mu+\eta}(\rho + \mu + \tau).$$

The summands are zero unless $\eta, \tau - \eta \in \Lambda$. To all terms with $\eta \neq \tau$ we can apply the induction hypothesis to the first factor and (4) to the second. Thus we obtain (with $\eta_i := \tau_i - \tau_{i-1}$)

$$a^h_\eta(\mu)p_{\mu+\eta}(\rho + \mu + \tau) =$$

$$= \sum_{\tau_*} \left[\sum_{i=0}^{r} \frac{(-1)^{r-i}}{(d-r)!i!(r-i)!} h(\rho+\tau_i) \right] f_{\eta_1}(\rho+\tau_1)\dots f_{\eta_d}(\rho+\tau_d)$$

where $r = \ell(\eta)$ and the sum runs over all paths τ_* from μ to $\mu+\tau$ with $\tau_r = \mu+\eta$. Thus we get

$$\sum_{\eta\neq\tau} a^h_\eta(\mu) p_{\mu+\eta}(\rho+\mu+\tau) =$$

$$= \sum_{\tau_*} \left[\sum_{r=0}^{d-1} \sum_{i=0}^{r} \frac{(-1)^{r-i}}{(d-r)!i!(r-i)!} h(\rho+\tau_i) \right] f_{\eta_1}(\rho+\tau_1)\dots f_{\eta_d}(\rho+\tau_d),$$

where we now sum over all paths from μ to $\mu+\tau$. Now we interchange the order of summation in the bracket:

$$\sum_{r=0}^{d-1} \sum_{i=0}^{r} \frac{(-1)^{r-i}}{(d-r)!i!(r-i)!} h(\rho+\tau_i) = \sum_{i=0}^{d-1} \left[\sum_{r=i}^{d-1} \frac{(-1)^{r-i}}{(d-r)!i!(r-i)!} \right] h(\rho+\tau_i).$$

The sum in brackets can be rewritten as

$$\sum_{r=i}^{d-1} \frac{(-1)^{r-i}}{(d-r)!i!(r-i)!} = \frac{1}{(d-i)!i!} \sum_{r=i}^{d-1} (-1)^{r-i} \binom{d-i}{r-i} = -\frac{(-1)^{d-i}}{(d-i)!i!}$$

Assembling everything together we get

$$a^h_\tau(\mu) = h(\rho+\tau_d) p_\mu(\rho+\tau_d) + \sum_{\tau_*} \left[\sum_{i=0}^{d-1} \frac{(-1)^{d-i}}{i!(d-i)!} h(\rho+\tau_i) \right] f_{\eta_1}(\rho+\tau_1)\dots f_{\eta_d}(\rho+\tau_d).$$

By (4), the first summand is nothing but the missing case $i = d$ of the second one. This yields (8). □

4.3. Corollary. *Let $\rho \in V_0$ be dominant. Let $\tau \in \Lambda$ and $\mu \in \Lambda_+$ with $\mu+\tau \in \Lambda_+$. Then*

$$a^h_\tau(\mu) = \sum_\eta (-1)^{\ell(\tau-\eta)} h(\rho+\mu+\eta) p_\mu(\rho+\mu+\eta) p_{\mu+\eta}(\rho+\mu+\tau)$$

where the sum runs over all $\eta \in \Lambda$ with $\tau-\eta \in \Lambda$ and $\mu+\eta \in \Lambda_+$.

Proof: By continuity, we may assume that ρ is non-integral. In (8), we fix an i and sum over all different $\eta := \tau_i$ first. Then the formula follows from two applications of Theorem 3.8. □

5. Construction of other difference operators

We are going to need the following

5.1. Lemma. *Let $\tau \in \Lambda$ and $\omega \in \Phi$ with $\omega(\tau) < 0$. Then $-\omega \in \Phi$.*

Proof: Since $\tau \in \Lambda$ there is $\eta \in \Lambda_1$ with $\omega(\eta) < 0$. By definition, there is a $w \in W$ with $w\eta \in \Sigma$. Then $w^{-1}\omega \notin \Phi^+$ which implies that $-w^{-1}\omega$, hence $-\omega$ is in Φ. □

Now we can prove a cut-off property which is dual to that in Lemma 3.2.

5.2. Lemma. *Let $\rho \in V_0$ be regular dominant, $\mu \in \Lambda_+$, $\eta \in \Lambda_1$, and $\lambda := \mu + \eta$. Then $f_\eta(-\rho - \mu) = 0$ whenever $\lambda \notin \Lambda_+$.*

Proof: Since $\lambda \notin \Lambda_+$ there is $\omega \in \Sigma$ with $\omega(\lambda) = \omega(\mu) + \omega(\eta) < 0$. Hence, **C7** implies $\omega(\mu) = 0$ and $\omega(\eta) = -1$. Since Lemma 5.1 implies $\bar{\omega} := -\omega \in \Phi$, the factor $\bar{\omega}(z) - k_{\bar{\omega}} = -\omega(z) - \omega(\rho)$ of $f_\eta(z)$ vanishes at $z = -\rho - \mu$. □

For $d \in \mathbb{N}$ we define the falling factorial functions as

$$[z \downarrow d] := z(z-1)\ldots(z-d+1).$$

Now we generalize the definition of f_τ to all $\tau \in \Lambda$ as follows:

$$f_\tau(z) := \frac{\prod_{\omega \in \Phi : \omega(\tau) > 0} [\omega(z) - k_\omega \downarrow \omega(\tau)]}{\prod_{\alpha \in \Delta : \alpha(\tau) > 0} [\alpha(z) \downarrow \alpha(\tau)]}. \tag{9}$$

Now we need stronger non-degeneracy conditions for ρ.

Definition: An element $\rho \in V_0$ is called *strongly dominant* if ρ is regular dominant and $\omega(\rho) - k_\omega \notin \mathbb{Z}_{<0}$ and $\omega(\rho) + k_\omega \notin \mathbb{Z}_{\leq 0}$ for all $\omega \in \Phi^+$.

The set of strongly dominant ρ is Zariski-dense in V_0. In fact, it suffices to show that it is non-empty since then it is the complement of countably many hyperplanes. To produce a strongly dominant ρ, we set all k_ω equal t where $t \notin \mathbb{Q}_{\leq 0}$. Every $\alpha \in \Delta^+$ and $\omega \in \Phi^+$ has an expression $\sum_{\omega_i \in \Sigma} a_i\omega_i$ with all $a_i \in \mathbb{Z}_{\geq 0}$. Let $N := \sum_i a_i \in \mathbb{Z}_{>0}$. Then $\alpha(\rho) = Nt \notin \mathbb{Z}_{\leq 0}$, $\omega(\rho) + k_\omega = (N+1)t \notin \mathbb{Z}_{\leq 0}$, or $\omega(\rho) - k_\omega = (N-1)t \notin \mathbb{Z}_{<0}$.

In fact, one can show that for the examples coming from multiplicity free spaces ρ is strongly dominant whenever all k_ω are real and positive.

5.3. Lemma. *Let $\rho \in V_0$ be strongly dominant and $\lambda, \mu \in \Lambda_+$, $\tau := \lambda - \mu$. Then $f_\tau(\rho + \lambda)$ and $f_\tau(-\rho - \mu)$ are defined and non-zero.*

Proof: Let $\alpha \in \Delta$ with $\alpha(\tau) > 0$. Consider the factor $F = [\alpha(\rho + \lambda) \downarrow \alpha(\tau)] = \big(\alpha(\rho) + \alpha(\lambda)\big)\ldots\big(\alpha(\rho) + \alpha(\mu) + 1\big)$. If $\alpha \in \Delta^+$, then $\alpha(\lambda) \geq \alpha(\mu) + 1 > 0$. From $\alpha(\rho) \notin \mathbb{Z} \leq 0$ we get $F \neq 0$. If $-\alpha \in \Delta^+$, then $0 \geq \alpha(\lambda) \geq \alpha(\mu) + 1$ and $\alpha(\rho) \notin \mathbb{Z}_{\geq 0}$ which again implies $F \neq 0$. Therefore, $f_\tau(\rho + \lambda)$ is defined.

The denominator of $f_\tau(-\rho - \mu)$ as well as the numerators are treated analogously by considering the factors

$$[\alpha(-\rho - \mu) \downarrow \alpha(\tau)] = \pm\big(\alpha(\rho) + \alpha(\lambda) - 1\big)\ldots\big(\alpha(\rho) + \alpha(\mu)\big),$$

$$[\omega(\rho+\lambda)-k_\omega\downarrow\omega(\tau)]=\big(\omega(\rho)-k_\omega+\omega(\lambda)\big)\ldots\big(\omega(\rho)-k_\omega+\omega(\mu)+1\big),$$
$$[\omega(-\rho-\mu)-k_\omega\downarrow\omega(\tau)]=\pm\big(\omega(\rho)+k_\omega+\omega(\lambda)-1\big)\ldots\big(\omega(\rho)+k_\omega+\omega(\mu)\big).\qquad\square$$

In particular, for every $\lambda\in\Lambda_+$ we can define the *virtual dimension* as

$$d_\lambda:=(-1)^{\ell(\lambda)}\frac{f_\lambda(-\rho)}{f_\lambda(\rho+\lambda)}\neq 0.$$

The terminology comes from the fact that, for multiplicity free spaces, d_λ is indeed the dimension of the simple module with highest weight λ. This will be proved in a forthcoming paper. In general, the virtual dimension can be rewritten as

$$d_\lambda=\prod_{\alpha\in\Delta^+}\frac{\alpha(\rho+\lambda)}{\alpha(\rho)}\prod_{\omega\in\Phi^+}\frac{[\omega(\rho+\lambda)+k_\omega-1\downarrow\omega(\lambda)]}{[\omega(\rho+\lambda)-k_\omega\downarrow\omega(\lambda)]}.$$

The first factor is just Weyl's dimension formula. It is easy to see that d_λ is a polynomial function in λ if and only if $k_\omega\in\frac{1}{2}\mathbb{Z}_{>0}$ for all ω. In this case we have

$$d_\lambda=\prod_{\alpha\in\Delta^+}\frac{\alpha(\rho+\lambda)}{\alpha(\rho)}\prod_{\omega\in\Phi^+}\prod_{s=-k_\omega+1}^{k_\omega-1}\frac{\omega(\rho+\lambda)+s}{\omega(\rho)+s}.$$

Another property of the virtual dimension is:

5.4. Theorem. *Let ρ be strongly dominant. Let $\lambda,\mu\in\Lambda_+$ with $\tau=\lambda-\mu\in\Lambda$. Then*

$$\frac{d_\lambda}{d_\mu}=(-1)^{\ell(\tau)}\frac{f_\tau(-\rho-\mu)}{f_\tau(\rho+\lambda)}.\tag{10}$$

Proof: With

$$A_\alpha:=\frac{[-\alpha(\rho+\mu)\downarrow\alpha(\tau)]}{[\alpha(\rho+\lambda)\downarrow\alpha(\tau)]},\quad B_\omega:=\frac{[-\omega(\rho+\mu)-k_\omega\downarrow\omega(\tau)]}{[\omega(\rho+\lambda)-k_\omega\downarrow\omega(\tau)]},$$

we get

$$\frac{f_\tau(-\rho-\mu)}{f_\tau(\rho+\lambda)}=\prod_{\alpha\in\Delta:\alpha(\tau)>0}A_\alpha^{-1}\prod_{\omega\in\Phi:\omega(\tau)>0}B_\omega.\tag{11}$$

It is customary to extend the definition of the falling factorial functions as $[z\downarrow d]:=1/(z+1)\ldots(z-d)$ when $d<0$. With this convention, the formula $[z\downarrow d]=1/[z-d\downarrow -d]$ holds for all $d\in\mathbb{Z}$ if the left-hand side is non-zero. This holds in our case and we obtain

$$\frac{[-\omega(\rho+\mu)-k_\omega\downarrow\omega(\tau)]}{[\omega(\rho+\lambda)-k_\omega\downarrow\omega(\tau)]}=\frac{[\omega(\rho+\mu)-k_\omega\downarrow-\omega(\tau)]}{[-\omega(\rho+\lambda)-k_\omega\downarrow-\omega(\tau)]}.$$

Now I claim that

$$\Phi_{\tau>0}=\Phi^+_{\tau>0}\cup(-\Phi^+_{\tau<0})$$

where the subscript $\tau > 0$ means "the subset of all ω with $\omega(\tau) > 0$", etc. If $\omega \in \Phi$, then either $\omega \in \Phi^+$ or $-\omega \in \Phi^+$ (by **C3**) which shows the inclusion "$\subseteq$". Conversely, let $\omega \in \Phi^+$ with $\omega(\eta) < 0$. Then $-\omega \in \Phi_{\tau>0}$ by Lemma 5.1, which proves the claim.

We have $k_\omega = k_{-\omega}$ and therefore $B_\omega = B_{-\omega}$ whenever both $\pm\omega \in \Phi$. Thus, the claim implies that in (11) we can replace the product over all $\omega \in \Phi_{\tau>0}$ by the product over all $\omega \in \Phi^+$. Similarly, we can change the range of the first product to Δ^+.

Now we apply the formulas

$$[z \downarrow a - b] = \frac{[z + b \downarrow a]}{[z + b \downarrow b]} = \frac{[z \downarrow a]}{[z - (a - b) \downarrow b]}$$

with $a = \omega(\lambda)$ and $b = \omega(\mu)$. They hold whenever none of the denominators are zero. That this is so in our case follows from Lemma 5.3. We obtain

$$B_\omega = \frac{[-\omega(\rho) - k_\omega \downarrow \omega(\lambda)]}{[-\omega(\rho) - k_\omega \downarrow \omega(\mu)]} \frac{[\omega(\rho + \mu) - k_\omega \downarrow \omega(\mu)]}{[\omega(\rho + \lambda) - k_\omega \downarrow \omega(\lambda)]}.$$

Similarly, we have

$$A_\alpha = \frac{[-\alpha(\rho) \downarrow \alpha(\mu)]}{[-\alpha(\rho) \downarrow \alpha(\lambda)]} \frac{[\alpha(\rho + \lambda) \downarrow \alpha(\lambda)]}{[\alpha(\rho + \mu) \downarrow \alpha(\mu)]}$$

for all $\alpha \in \Delta^+$. Since $\lambda, \mu \in \Lambda_+$ we can multiply in the definition (9) of f_λ, f_μ over all $\alpha \in \Delta^+$ and $\omega \in \Phi^+$. The asserted formula (10) follows readily. $\square$

For $\tau \in \Lambda$ with $d := \ell(\tau)$ we now define the rational function

$$\text{(12)} \qquad b^h_\tau(z) := \sum_{\tau_*} \left[\sum_{i=0}^{d} \frac{(-1)^{d-i}}{i!(d-i)!} h(z - \tau_i) \right] f_{\tau_1 - \tau_0}(z - \tau_0) \dots f_{\tau_d - \tau_{d-1}}(z - \tau_{d-1})$$

where the sum runs over all paths from 0 to τ. For $h \in \mathcal{P}$ define $h^- \in \mathcal{P}$ by $h^-(z) = h(-z)$.

5.5. Theorem. *Let ρ be strongly dominant and non-integral. Let $\lambda, \mu \in \Lambda_+$ with $\tau = \lambda - \mu \in \Lambda$. Then*

$$\text{(13)} \qquad b^h_\tau(-\rho - \mu) = (-1)^{\ell(\tau)} \frac{d_\lambda}{d_\mu} a^{h^-}_\tau(\mu).$$

Proof: This would follow directly from Theorem 4.2 and Theorem 5.4 if in (12) the summation were restricted to those paths τ_* for which $\mu + \tau_*$ is positive. But other paths do not contribute to $b^h_\tau(-\rho - \mu)$ anyway: let i be minimal such that $\mu + \tau_i \notin \Lambda_+$. Then $f_{\eta_i}(-\rho - \mu - \tau_{i-1}) = 0$, by Lemma 5.2. $\square$

The following consequence is crucial:

5.6. Corollary. *Let $\tau \in \Lambda$ with $\ell(\tau) > \deg h$. Then $b^h_\tau(z) = 0$.*

Proof: For strongly dominant, non-integral ρ this follows from Theorem 5.5 and Proposition 4.1 since the set of points $z = -\rho - \mu$ with $\mu \in \Lambda_+$ and $\tau + \mu \in \Lambda_+$ is Zariski dense. For general ρ we conclude by continuity. $\square$

Thus, for each $h \in \mathcal{P}^W$ we can define the difference operator

$$D_h := \sum_{\tau \in \Lambda} b_\tau^h(z) T_\tau.$$

We are going to rewrite it: Fix $\tau \in \Lambda$ and put $i := \ell(\tau)$. Then (with $\eta_i := \tau_i - \tau_{i-1}$)

$$\sum_{\tau_*, \tau_i = \tau} h(z - \tau_i) f_{\eta_1}(z - \tau_0) \ldots f_{\eta_d}(z - \tau_{d-1}) =$$

$$= \sum_{\tau_*, \tau_i = \tau} (T_{\eta_1} \ldots T_{\eta_i})(h) \cdot f_{\eta_1} \cdot T_{\eta_1}(f_{\eta_2}) \cdot (T_{\eta_1} T_{\eta_2})(f_{\eta_3}) \cdot \ldots \cdot (T_{\eta_1} \ldots T_{\eta_{d-1}})(f_{\eta_d}).$$

This is easily recognized as the coefficient of T_τ in $L^i h L^{d-i}$ where we regard h as a multiplication operator. Thus we get

$$\sum_{\ell(\tau) = d} b_\tau^h(z) T_\tau = \frac{1}{d!} \sum_{i=0}^{d} (-1)^{d-i} \binom{d}{i} L^i h L^{d-i} = \frac{1}{d!} (\operatorname{ad} L)^d(h).$$

Using Corollary 5.6 we get

5.7. Corollary. *Let $h \in \mathcal{P}^W$ and $d \in \mathbb{N}$ with $d > \deg h$. Then* $(\operatorname{ad} L)^d(h) = 0$.

We also obtained the first part of:

5.8. Theorem. *a) Let $h \in \mathcal{P}^W$. Then $D_h = \exp(\operatorname{ad} L)(h)$.*
b) The map $h \mapsto D_h$ is an algebra homomorphism. In particular, the D_h commute pairwise.
c) If ρ is dominant, then the action of the D_h on $\mathcal{P}^W$ is simultaneously diagonalizable. More precisely, if $\lambda \in \Lambda_+$ then $D_h(p_\lambda) = h(\rho + \lambda) p_\lambda$.

Proof: Let $\mathcal{A} \subseteq \operatorname{End}(\mathcal{P}^W)$ be the largest subalgebra on which $\operatorname{ad} L$ acts locally nilpotently. Then $\mathcal{P}^W \subseteq \mathcal{A}$. Moreover, $\operatorname{ad} L$ is a derivation, hence $\exp(\operatorname{ad} L)$ is an automorphism of $\mathcal{A}$. This shows *b)*.

In *c)*, we may assume that ρ is regular dominant and then conclude by continuity. First, observe $E = D_\ell$. Therefore D_h and E commute. The space of $f \in \mathcal{P}^W$ with $\deg f \le e$ can be characterized as the direct sum of the E-eigenspaces for the eigenvalues $\ell(\rho), \ell(\rho) + 1, \ldots, \ell(\rho) + e$. Therefore, $D_h(f) \le \deg f$ for all $f \in \mathcal{P}^W$. On the other hand, both L and h, hence D_h, preserve the space M_λ from Corollary 3.3. We conclude that $D_h(p_\lambda) = c p_\lambda$ for some constant c. The constant term of D_h, i.e., the coefficient of T_0, is h. Thus, evaluating at $z = \rho + \lambda$ gives $c = h(\rho + \lambda)$. This shows c). $\square$

6. Further analysis of the difference operators

In this section, we derive some basic properties of the functions $b^h_\tau(z)$. First the degree:

6.1. Proposition. *Let $h \in \mathcal{P}^W$ and $\tau \in \Lambda$. Then* $\deg b^h_\tau(z) \le \deg h$.

Proof: Let $c(z)$ be a rational function of degree d on V and $\tau \in \Gamma$. Then $[L, cT_\tau] = [L, c]T_\tau + c[L, T_\tau]$. We have $[L, c] = \sum_\eta f_\eta(z)(c(z-\tau) - c(z))$. Since $\deg f_\eta(z) \le 1$ and $\deg(c(z-\tau) - c(z)) < d$ we see that $[L, c]T_\tau$ has coefficients of degree $\le d$. Similarly for the other term: The coefficients in $[L, T_\tau] = \sum_\eta c(z)(f_\eta(z) - f_\eta(z))T_{\tau+\eta}$ have degree $\le d$. Thus we have shown that $\operatorname{ad} L$ does not increase the degrees of coefficients. The assertion follows from the formula Theorem 5.8a). $\square$

Next we study the denominator:

6.2. Proposition. *Let $h \in \mathcal{P}^W$. For fixed $\tau \in \Lambda$ with $b^h_\tau \neq 0$ and $\alpha \in \Delta$ put*

$$S(\alpha, \tau) := \{i \in \mathbb{Z} \mid i \neq 0, b^h_{\tau + i\alpha^\vee} \neq 0\}.$$

Then

$$b^h_\tau(z) \prod_{\alpha \in \Delta^+} \prod_{i \in S(\alpha,\tau)} (\alpha(z-\tau) - i)$$

is a polynomial in z.

Proof: By the explicit formula in Theorem 5.8*a)*, the only denominators which can occur are products of terms $\alpha(z) - m$ with $\alpha \in \Delta$ and $m \in \mathbb{Z}$. Fix one such factor $\alpha(z) - m$ and let S be the set of all τ such that $\alpha(z) - m$ occurs in the denominator of b_τ. Let $\mathcal{H}_0$ be the hyperplane $\{\alpha - m = 0\}$.

Fix $\tau \in S$ and let $z \in \mathcal{H}_0$. Suppose that none of the points $z - \tau'$, $\tau \in S$, $\tau' \neq \tau$ is in the W-orbit of $z - \tau$. Then one could find a *symmetric* function f which does not vanish in $z - \tau$ but vanishes to a very high order in all other points $z - \tau'$. Then $D_h(f)$ would not be regular in z. Thus, there must be τ' and w with $w(z - \tau') = z - \tau$. Since there are only finitely many choices of w and τ' there is one choice which works for a dense subset of points $z \in \mathcal{H}_0$. By continuity, we get

$$w(z - \tau') = z - \tau \text{ for all } z \in \mathcal{H}_0. \tag{14}$$

Choose any $z_0 \in \mathcal{H}_0$. Then for any y with $\alpha(y) = 0$ and any number t we have $z = t * y + z_0 \in \mathcal{H}_0$. Comparing the coefficient of t in (14) yields $w(y) = y$. Since w cannot be the identity we get $w = s_\alpha$. Also τ' is unique since

$$\tau' = z - s_\alpha(z - \tau) = \tau + (m - \alpha(\tau))\alpha^\vee.$$

This yields $\alpha(z) - m = \alpha(z - \tau) - i$ with $i = m - \alpha(\tau) \in S(\alpha, \tau)$.

It remains to show that $\alpha(z) - m$ occurs with multiplicity one in the denominator of b_τ^h. Let $N > 0$ be the larger of the powers with which $\varepsilon := \alpha(z) - m$ occurs in the denominator of b_τ^h and $b_{\tau'}^h$. Put $c := \varepsilon^N b_\tau^h$ and likewise $c' := \varepsilon^N b_{\tau'}^h$. Then for every $f \in \mathcal{P}^W$, the rational function $c(z)f(z-\tau) + c'(z)f(z-\tau')$ has a zero of order at least N along the divisor $\varepsilon(z) = 0$. From the identity $z - \tau' = s_\alpha(z - \tau - \varepsilon\alpha^\vee)$ we infer that

$$c(z)f(z-\tau) + c'(z)f(z-\tau') = \tag{15}$$

$$= (c(z) + c'(z))f(z-\tau) + c'(z)(f(z-\tau-\varepsilon\alpha^\vee) - f(z-\tau)).$$

We conclude that $c(z)+c'(z)$ is divisible by ε. Hence, since one of c or c' is not divisible by ε the other is not either. Next observe that $\tau \neq \tau'$ implies that $\mathcal{H}_0 - \tau$ is not the reflection hyperplane of s_α. Hence there exists $z \in cH_0$ and $f \in \mathcal{P}^W$ which is divisible by ε such that the directional derivative $D_{\alpha^\vee} f(z-\tau) \neq 0$. This implies that the right-hand side of (15) vanishes to exactly first order in $\mathcal{H}_0$. Thus $N = 1$. □

A lower bound for the numerator is given by

6.3. Proposition. *Let ρ be non-integral. Then the numerator of b_τ^h is divisible by*

$$\prod_{\omega\in\Phi:\omega(\tau)>0} [\omega(z) - k_\omega \downarrow \omega(\tau)].$$

Proof: The non-integrality of ρ ensures that the denominator of $b_\tau^h(z)$ does not vanish whenever $z \in \rho + \Gamma$. If $\omega \in \Sigma$, then the definition of $b_\tau^h(z)$ along with Lemma 5.2 implies that $b_\tau^h(z) = 0$ for all $z = \rho + \lambda$ with $\lambda \in \Lambda_+$ and $\omega(\lambda - \tau) < 0$. Thus, b_τ^h is divisible by $[\omega(z) - k_\omega \downarrow \omega(\tau)]$.

Let $\omega \in \Phi$ be arbitrary with $\omega(\tau) > 0$. There is $w \in W$ with $w\omega \in \Sigma$. Thus, by the case above, $b_{w\tau}^h(z)$ is divisible by $[w\omega(z) - k_\omega \downarrow \omega(\tau)]$. The rest follows from the fact that D_h is a symmetric operator: $b_{w\tau}^h(wz) = b_\tau(z)$. □

We now introduce an order relation on Λ: We define $\tau_1 \leq \tau_2$ if $\ell(\tau_1) < \ell(\tau_2)$ or $\ell(\tau_1) = \ell(\tau_2)$ and $\tau_2 - \tau_1$ is a sum of positive roots.

6.4. Theorem. *For $h \in \mathcal{P}^W$ let $\tau \in \Lambda_+$ be maximal (with respect to "$\leq$") with $b_\tau^h \neq 0$. Then $\ell(\tau) = \deg h$, $\tau \in \Lambda_+$ and $b_\tau^h(z) \in \mathbb{C}^* f_\tau(z)$.*

Proof: First of all, it follows from Proposition 4.1 and Theorem 5.5 that $\ell(\tau) = \deg h$. The set of τ with $b_\tau^h \neq 0$ is W-stable. It follows that maximality of τ implies $\alpha(\tau) \geq 0$ for all $\alpha \in \Delta^+$. From $s_\alpha(\tau) = \tau - \alpha(\tau)\alpha^\vee$ it follows that $i = -\alpha(\tau)$ is the minimal element of $S(\alpha, \tau)$ and all other i satisfy $-\alpha(\tau) \leq i < 0$. Thus Propositions 6.2 and 6.3 imply that $c(z) := b_\tau^h(z) f_\tau(z)^{-1}$ is a polynomial. Moreover, by Proposition 6.1 we have $\deg c(z) \leq \deg h - \deg f_\tau = \deg \ell(\tau) - \deg f_\tau = -j(\tau)$ where $j(\tau) := \deg f_\tau - \ell(\tau)$. Now all

assertions follow from the following claim: *Let $\tau \in \Lambda$ be dominant for Δ^+. Then $j(\tau) \geq 0$ and equality holds if and only if $\tau \in \Lambda_+$.*

To prove the claim let $a_+ := \max(a, 0)$ for any $a \in \mathbb{Z}$. Then, from the definition of f_τ it follows that

$$\deg f_\tau = \sum_{\omega \in \Phi} \omega(\tau)_+ - \sum_{\alpha \in \Delta^+} \alpha(\tau).$$

Thus $\tau \mapsto j(\tau)$ is a piecewise linear convex function. Moreover, $j(\tau) = 0$ whenever $\tau \in \Lambda_+$ (by **C3** and **C5**). Together this implies $j(\tau) \geq 0$ for all τ. Every $\omega_0 \in \Sigma$ defines a codimension-1-face of Λ_+. Let $\tau \in \Lambda$ be close enough to that face such that $\omega(\tau) \geq 0$ for all $\omega \in \Phi^+$ except for $\omega = \omega_0$ where $\omega_0(\tau) < 0$. Since $-\omega_0 \in \Phi$ by Lemma 5.1, **C5** then implies $j(\tau) = 0 - \omega_0(\tau) + (-\omega_0)(\tau) = -2\omega_0(\tau) > 0$. Therefore, $j(\tau)$ takes strictly positive values outside Λ_+ which finishes the proof of the claim. $\square$

6.5. Corollary. *Assume ρ is strongly dominant. For $\lambda \in \Lambda_+$ let $h = p_\lambda^-$. Then $b_\tau^h = 0$ unless $\tau \leq \lambda$ and $b_\lambda^h = f_\lambda(\rho + \lambda)^{-1} f_\lambda$.*

Proof: Let τ be maximal with $b_\tau^h \neq 0$. Then $\tau \in \Lambda_+$ and $b_\tau^h = c f_\tau$ for some $c \neq 0$. For $g := h^- = p_\lambda$ we get by definition $a_\tau^g(0) = \delta_{\lambda\tau}$. Theorem 5.5 implies

$$c f_\tau(-\rho) = b_\tau^h(-\rho) = (-1)^{\ell(\tau)} d_\tau \delta_{\lambda\tau} = \frac{f_\tau(-\rho)}{f_\tau(\rho + \tau)} \delta_{\lambda\tau}.$$

Hence Lemma 5.3 implies $\tau = \lambda$ and $c = f_\lambda(\rho + \lambda)^{-1}$. $\square$

With our Pieri formula we can convert this into a triangularity result. For this, it is convenient to introduce another normalization of the $p_\lambda(z)$: let $P_\lambda := f_\lambda(\rho + \lambda) p_\lambda$. Thus $P_\lambda(\lambda + \rho) = f_\lambda(\rho + \lambda)$.

6.6. Corollary. *For all $\lambda, \mu \in \Lambda_+$ holds*

$$P_\lambda P_\mu = P_{\lambda+\mu} + \sum_{\nu < \lambda+\mu} c_{\lambda\mu}^\nu P_\nu.$$

Proof: For $h = P_\lambda^-$ we have just seen $b_\tau^h = 0$ unless $\tau \leq \lambda$ and $b_\lambda^h = f_\lambda$. Proposition 4.1 implies $P_\lambda P_\mu = \sum_\nu c_{\lambda\mu}^\nu P_\nu$ with $c_{\lambda\mu}^\nu = f_\lambda(\rho + \lambda) \frac{f_\mu(-\rho)}{f_\nu(-\rho)} b_{\nu-\mu}^h(-\rho - \mu)$. Thus $c_{\lambda\mu}^\nu = 0$ unless $\nu - \mu \leq \lambda$. Moreover,

$$c_{\lambda\mu}^{\lambda+\mu} = \frac{f_\lambda(-\rho - \mu) f_\mu(-\rho)}{f_{\lambda+\mu}(-\rho)}.$$

This last expression equals 1 as follows from the identity $[z - b \downarrow a][z \downarrow b] = [z \downarrow a + b]$. $\square$

In the classical and semiclassical case, P_λ is exactly the polynomial obtained by normalizing the leading coefficient to 1. To make sense of this in general we introduce a monomial basis of $\mathcal{P}^W$ as follows: consider $\Sigma = \{\omega_1, \ldots, \omega_r\}$ and its dual basis $\Sigma^\vee = \{\eta_1, \ldots, \eta_r\}$. Then for any $\lambda \in \Lambda_+$ we define

$$\mathbf{e}_\lambda := \prod_{i=1}^{r} P_{\eta_i}^{\omega_i(\lambda)}.$$

Then we get easily by induction:

6.7. Corollary. *For every $\lambda \in \Lambda_+$ there is an expansion*

$$P_\lambda = \mathbf{e}_\lambda + \sum_{\mu<\lambda} d_{\lambda\mu}\mathbf{e}_\mu.$$

7. Multiplicity free spaces

In this section, we introduce the main class of examples to which the theory developed in the preceding sections applies.

Let G be a connected reductive group (everything is defined over $\mathbb{C}$) and U a finite dimensional G-module. Let $\mathcal{O}(U)$ be its algebra of polynomial functions. Then U is called a *multiplicity free space* if $\mathcal{O}(U)$ is a multiplicity free G-module, i.e., every simple G-module occurs in $\mathcal{O}(U)$ at most once. A more geometric criterion is due to Vinberg-Kimelfeld ([VK], see also [Kn1] Thm. 3.1): a Borel subgroup of G has a dense orbit in U.

We assume from now on that U is a multiplicity free space and we are going to derive a structure $(\Gamma, \Sigma, W, \ell)$ from it. For a dominant integral weight λ let M^λ be the simple G-module with *lowest* weight $-\lambda$. Let Λ_+ be the set of λ such that M^λ occurs in $\mathcal{O}(U)$. Thus, as a G-module, we have

$$\mathcal{O}(U) \cong \oplus_{\lambda\in\Lambda_+} M^\lambda.$$

We regard characters as elements of the dual Cartan algebra $\mathfrak{t}^*$. Let $\Gamma^\vee \subseteq \mathfrak{t}^*$ be the subgroup generated by Λ_+. Then we can define the first ingredient as $\Gamma := \mathrm{Hom}(\Gamma^\vee, \mathbb{Z})$.

It is known ([HU], see also [Kn1] Thm. 3.2) that Γ_+ is a monoid which is generated by a basis $\Sigma^\vee$ of $\Gamma^\vee$. Let $\Sigma \subseteq \Gamma$ be its dual basis. Since U is a vector space, the algebra $\mathcal{O}(U)$ is graded and every irreducible constituent M^λ occurs in some degree $\ell(\lambda)$. It is easy to see that ℓ is additive on Λ_+. Hence it extends to a linear function $\ell \in \Gamma$.

The reflection group W is more involved to construct. Let $\mathcal{D}(U)$ be the algebra of polynomial coefficient differential operators on U. We are interested in the algebra of G-invariant operators $\mathcal{D}(U)^G$. Every $D \in \mathcal{D}(U)^G$ acts on $M^\lambda \subseteq \mathcal{O}(U)$ by a scalar, denoted by $c_D(\lambda)$. Let $V \subseteq \mathfrak{t}^*$ be the $\mathbb{C}$-span of $\Gamma^\vee$. By [Kn1] Cor. 4.4, the function $c_D(\lambda)$

extends to a polynomial function $c_D(z)$ on V. Thus we get an injective homomorphism $c : \mathcal{D}(U)^G \hookrightarrow \mathcal{O}(V) : D \mapsto c_D$.

To determine its image let $\bar\rho \in \mathfrak{t}^*$ be the half-sum of the positive roots and $\overline{W} \subseteq GL(\mathfrak{t}^*)$ the Weyl group of G. The twisted action of $\overline{W}$ on $\mathfrak{t}^*$ is defined as $w \bullet \chi := w(\chi + \bar\rho) - \bar\rho$. Then W is characterized as follows:

7.1. Theorem. *There is a unique subgroup $W \subseteq \overline{W}$ such that*

a) the subspace V and the lattice $\Gamma^\vee$ are stable under the twisted action of W;

b) the image of c consists exactly of the invariants under this twisted W-action.

This finishes the description of the structure $(\Gamma, \Sigma, W, \ell)$. The main point is the following theorem whose proof will occupy the rest of this section.

7.2. Theorem. *Let $(\Gamma, \Sigma, W, \ell)$ be the structure derived from a multiplicity free space. Then all axioms* **C1** *through* **C9** *hold.*

C1 follows from the fact that $\mathcal{D}(U)^G$ is a polynomial ring ([HU]; see also [Kn1] Cor. 4.7) and the Shepherd-Todd theorem.

C2 is clear since all weights in $\Sigma^\vee$ are dominant.

C4 follows from Theorem 7.1 since the Euler vector field ξ is in $\mathcal{D}(U)^G$ and we have $c_\xi = \ell$.

C6 is trivial, since degrees of non-constant polynomials are strictly positive.

C8 Any linear W-invariant f comes from a G-invariant differential operator of order one, hence from a G-invariant vector field ξ on U. We have $U^\vee \subseteq \mathcal{O}(U)$ and $\xi(U^\vee) \subseteq U^\vee$. Thus, ξ is uniquely determined by the G-endomorphism $\xi_0 := \xi|_U$. This endomorphism ξ_0 acts on each simple component of lowest weight η of U as scalar $f(\eta)$. But these lowest weights run exactly through $\Sigma_1^\vee$.

C3, **C5**, and **C7** are handled case by case. For this, we first need some reductions. Assume that there is a reductive group $\overline{G}$ with $G \subseteq \overline{G} \subseteq GL(U)$ such that G is normal in $\overline{G}$ and the quotient $\overline{G}/G$ is a torus. Then the center $\overline{Z}$ of $\overline{G}$ acts as a scalar on each module $M^\lambda \subseteq \mathcal{O}(U)$. This means that U is also multiplicity free with respect to $\overline{G}$ and that there is an isomorphism $\overline{\Lambda}_+ = \Lambda_+$. A differential operator is in $\mathcal{D}(U)^G$ if and only if it acts as a scalar on each M^λ. This shows that $\mathcal{D}(U)^G = \mathcal{D}(U)^{\overline{G}}$. Hence we obtain an isomorphism $(\Gamma, \Sigma, W, \ell) \cong (\overline{\Gamma}, \overline{\Sigma}, \overline{W}, \bar\ell)$.

This observation is applied as follows: let $U = U_1 \oplus \ldots \oplus U_s$ be the decomposition of U into simple modules and let $A = \mathbf{G}_m^s \subseteq GL(U)$ consisting of the scalar multiplications in each factor. Then we can replace G by $\overline{G} = AG$, i.e., assume right away that $A \subseteq G$. A multiplicity free space with that property is called *saturated*.

If (G_1, U_1) and (G_2, U_2) are multiplicity free spaces then $(G, U) = (G_1 \times G_2, U_1 \oplus U_2)$ is one as well. If U_1 and U_2 are non-zero, then U is called *decomposable*. The combinatorial structures are related by $(\Gamma, \Sigma, W, \ell) = (\Gamma_1 \oplus \Gamma_2, \Sigma_1 \cup \Sigma_2, W_1 \times W_2, \ell_1 + \ell_2)$. Moreover, one

readily verifies that all axioms, but in particular **C3**, **C5**, and **C7**, hold for U whenever they hold for U_1 and U_2.

Thus we may assume that U is indecomposable and saturated. These multiplicity free spaces have been classified independently by Benson-Ratcliff, [BR1], and Leahy, [Le]. This classification together with the structure $(\Gamma, \Sigma, W, \ell)$ is tabulated in the next section from which one easily verifies the three axioms case by case.

Finally, we verify axiom **C9**. This will also provide the motivation for the whole theory. Observe, that, with U, the dual representation $U^\vee$ is also a multiplicity free space. Its algebra of functions decomposes as a G-module like

$$\mathcal{O}(U^\vee) = \oplus_{\lambda\in\Lambda_+} M_\lambda$$

where $M_\lambda = (M^\lambda)^\vee$ is the simple G-module with *highest* weight λ. An element $D \in \mathcal{O}(U^\vee)$ can be regarded as a differential operator with constant coefficients on U. Let $\mathcal{D}(U)$ be the algebra of all polynomial coefficient differential operators. Thus we get a G-module isomorphism

$$\mathcal{O}(U) \otimes \mathcal{O}(U^\vee) \to \mathcal{D}(U) : f \otimes D \mapsto fD.$$

Using this isomorphism we can construct a distinguished basis of $\mathcal{D}(U)^G$ as follows: the space of G-fixed vectors in $M^\lambda \otimes M_\mu$ is non-zero if and only if $\lambda = \mu$ and in that case it is one-dimensional. Let $D_\lambda \in \mathcal{D}(U)^G$ be the image of a generator. Then the family D_λ, $\lambda \in \Lambda_+$, is a basis of $\mathcal{D}(U)^G$. Thus, the polynomials $c_\lambda(z) := c_{D_\lambda}(z)$ form a basis of the space of shifted invariant polynomials on V. To get rid of the shift, choose any W-equivariant projection $\pi : \mathfrak{t}^* \to V$ and any vector $\sigma \in V^W$. Let $\kappa := \bar\rho - \pi(\bar\rho)$ and $\rho := \pi(\bar\rho) + \sigma = \bar\rho - \kappa + \sigma \in V$. Then the fact that $\bar\rho + V$ is W-stable means precisely that κ is W-fixed. Hence the shifted W-action with $\bar\rho$-shift coincides with that corresponding to the ρ-shift. Thus, if we put $p_\lambda(z) := c_\lambda(z-\rho)$ then we obtain a basis of the (unshifted) W-invariants on V. Now, condition **C9** is essentially Theorem 4.10 of [Kn1]. That theorem also makes sure that we can normalize D_λ in such a way that $p_\lambda(\rho+\lambda) = c_\lambda(\lambda) = 1$.

The only point left to show is that π and σ can be chosen in such a way that $\rho \in V_0$. We are even going to construct a canonical element $\rho \in V_0$.

Recall the following consequence of the local structure theorem (see, e.g., [Kn1] Thm. 2.4). Let B^- be a Borel subgroup opposite to B. Then there is a point $u \in U$ such that B^-u is open in U. Moreover, there is a parabolic subgroup $P \subseteq G$ with Levi part L and unipotent radical R_uP such that

- the orbit $Pu = B^-u$;
- the isotropy group P_u is contained and normal in L;
- the quotient L/P_u is a torus.

For $\lambda \in \Lambda_+$, the lowest weight vector of $f_\lambda \in M^\lambda$ can be normalized to $f_\lambda(u) = 1$ and then defines a character of P, hence of L. The intersection of the kernels of these characters equals $L_u = P_u$. Therefore, we can identify V with the dual of $\operatorname{Lie} L/P_u$. Furthermore, we can choose a subspace C of the center of $\operatorname{Lie} L$ such that $\operatorname{Lie} L = C \oplus \operatorname{Lie} P_u$ and $V \cong C^*$. For a Cartan subalgebra $\mathfrak{t}$ of $\operatorname{Lie} L$ we obtain $\mathfrak{t}^* = C^* \oplus \mathfrak{t}_u^*$. Now we choose for π the projection of $\mathfrak{t}^*$ onto $C^* = V$.

Let w_L be the longest element of the Weyl group of L with respect to $\mathfrak{t}$. Then clearly $\pi(\bar\rho) = \pi(w_L\bar\rho)$, i.e., $\tilde\rho := \frac{1}{2}(\bar\rho + w_L\bar\rho)$ has the same image in V as $\bar\rho$. Let $\chi \in \mathfrak{t}^*$ be sum of all weights of U. Since it comes from a character of $\operatorname{Lie} G$, it is $\overline{W}$-invariant. Let $\rho := \frac{1}{2}\chi + \tilde\rho = \frac{1}{2}(\chi + \bar\rho + w_L\bar\rho)$. Then $\pi(\rho)$ differs from $\pi(\bar\rho)$ by the W-invariant element $\sigma := \frac{1}{2}\pi(\chi)$.

Now I claim that $\pi(\rho) = \rho$, i.e., ρ is already in V. Consider the action of $\mathfrak{t}_u$ on the top exterior power of the tangent space $\Lambda^{\mathrm{top}} T_u U = \Lambda^{\mathrm{top}} U$. On one side, this is just the restriction of χ to $\mathfrak{t}_u$. On the other side, observe that, $-2\tilde\rho$ is the sum of all roots belonging to $R_u P$. Moreover, $T_u U = \operatorname{Lie} Pu = \operatorname{Lie} R_u P \oplus C$ with trivial action of $\mathfrak{t}_u$ on C. Thus, the restriction of ρ to $\mathfrak{t}_u$ is zero which means $\rho \in V$.

This element ρ is very easy to calculate in every given case. For the indecomposable saturated multiplicity free spaces it is recorded in the tables below. This shows in particular that $\rho \in V_0$ in every given case and concludes the proof of Theorem 7.2.

An immediate consequence is the following statement. It would be desirable to have a conceptual proof.

7.3. Corollary. *The polynomials p_λ describing the spectrum of Capelli operators on a multiplicity free space are the joint eigenfunctions of a family of commuting difference operators.*

8. Tables

The table below lists all structures $(\Gamma, \Sigma, W, \ell)$ which come from indecomposable saturated multiplicity free actions. First, the combinatorial structure is defined (indicated by a double line $||$) and then its relation to multiplicity free spaces.

The space V is a subspace of some $\mathbb{C}^m$ with canonical basis e_i and coordinates z_i. In all cases, except $\mathbf{III}_{\mathrm{odd}}$ and **IVa**, we have $V = \mathbb{C}^m$. In case **V**, we found it easier to work with basis vectors e_i, e_i', e'' and corresponding coordinates z_i, z_i', z''.

The Weyl group is given as follows: s_{ij} denotes the transposition $z_i \leftrightarrow z_j$. The notation $S_3(z_1, z_3, z_5)$ means the symmetric group permuting the coordinates z_1, z_3, z_5 and leaving all others fixed. A similar convention holds for other reflection groups, e.g., $\mathsf{D}_3(z_2, z_4, z_6)$. Finally, $\pm z_i$ means the reflection about the hyperplane $z_i = 0$.

The sets Δ^+ and Φ^+ are given such that condition **C5** can be verified easily.

We also give the orbit structure of Σ under the group $\pm W$. Each entry corresponds to an element of Σ. Then $\omega_i \in \pm W\omega_j$ if and only if the i-th and j-th entry are equal, disregarding the sign. The sign "$\pm$" stands if and only if $-\omega_i \in W\omega_i$. Otherwise, the same or different sign means that $\omega_i \in W\omega_j$ or $-\omega_i \in W\omega_j$, respectively.

Then we give the z-coordinates of a general element $\rho \in V_0$. If in the $\pm W$-table $\omega \in \Sigma$ has an upper-case letter R, S, etc. then $\omega(\rho)$ is denoted by the corresponding lower case letter r, s, etc.

Below the definition of the structure we list the indecomposable saturated multiplicity free actions giving rise to it. The list is comprehensive by the classification in [Kac], [BR1], and [Le]. In [Kn1] we collected all the necessary data to verify the assertions in the table.

In each case, we list the values of $\omega(\rho)$, $\omega \in \Sigma$ where $\rho = \frac{1}{2}(\chi + \bar\rho + w_L\bar\rho)$ is the canonical choice of ρ as described in the preceding section. Then we describe how the basis vectors e_i are related to actual weights. The notation is quite straightforward: ε_i denotes a weight in the defining representation of a classical group, α_i and ω_i are simple roots and fundamental weights (numbered as in Bourbaki [Bou]). Weights of different factors of G are distinguished by primes: ε_i, ε_i', ε_i'', etc.

Case I: The classical cases: $(1 \le n)$ (see also [KS])

$\Sigma := \{z_1 - z_2, z_2 - z_3, \dots, z_{n-1} - z_n, z_n\}$.
$\Sigma^\vee = \{e_1, e_1 + e_2, e_1 + e_2 + e_3, \dots, e_1 + e_2 + \dots + e_n\}$
$\ell := z_1 + z_2 + \dots + z_n$
$W := S_n = \langle s_{12}, s_{23}, \dots, s_{n-1\,n}\rangle$
$\Delta^+ = \{z_i - z_j \mid 1 \le i < j \le n\}$
$\Phi^+ = \{z_i - z_j \mid 1 \le i < j \le n\} \cup \{z_i \mid 1 \le i \le n\}$
$\pm W$-orbits of Σ: $[\pm R, \pm R, \dots, \pm R, S]$
$\rho = ((n-1)r + s, (n-2)r + s, \dots, r + s, s)$

$GL_p(\mathbb{C})$ on $S^2(\mathbb{C}^p)$ with $1 \le p$
$n = p$, $r = \frac{1}{2}$, $s = \frac{1}{2}$, $e_i = 2\varepsilon_i$

$GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$ on $\mathbb{C}^p \otimes \mathbb{C}^q$ with $1 \le p \le q$
$n = p$, $r = 1$, $s = \frac{1}{2}(q - p + 1)$, $e_i = \varepsilon_i + \varepsilon_i'$

$GL_p(\mathbb{C})$ on $\Lambda^2(\mathbb{C}^p)$ with $2 \le p$
p even: $n = \frac{p}{2}$, $r = 2$, $s = \frac{1}{2}$, $e_i = \varepsilon_{2i-1} + \varepsilon_{2i}$
p odd: $n = \frac{p-1}{2}$, $r = 2$, $s = \frac{3}{2}$, $e_i = \varepsilon_{2i-1} + \varepsilon_{2i}$

$Sp_{2p}(\mathbb{C})$ on $\mathbb{C}^{2p}$ with $1 \le p$
$n = 1$, r undefined, $s = p$, $e_1 = \varepsilon_1$

$SO_p(\mathbb{C}) \times \mathbb{C}^*$ on $\mathbb{C}^p$ with $3 \le p$
$n = 2$, $r = \frac{p}{2} - 1$, $s = \frac{1}{2}$, $e_1 = \varepsilon_1 + \varepsilon$, $e_2 = -\varepsilon_1 + \varepsilon$

$Spin_{10}(\mathbb{C}) \times \mathbb{C}^*$ on $\mathbb{C}^{16}$

$n = 2,\ r = 3,\ s = \frac{5}{2},\ e_1 = \omega_5 + \varepsilon,\ e_1 + e_2 = \omega_1 + 2\varepsilon$

$Spin_7(\mathbb{C}) \times \mathbb{C}^*$ on $\mathbb{C}^8$

$n = 2,\ r = 3,\ s = \frac{1}{2},\ e_1 = \omega_3 + \varepsilon,\ e_2 = -\omega_3 + \varepsilon$

$\mathrm{G}_2 \times \mathbb{C}^*$ on $\mathbb{C}^7$

$n = 2,\ r = \frac{5}{2},\ s = \frac{1}{2},\ e_1 = \omega_1 + \varepsilon,\ e_2 = -\omega_1 + \varepsilon$

$\mathrm{E}_6 \times \mathbb{C}^*$ on $\mathbb{C}^{27}$

$n = 3,\ r = 4,\ s = \frac{1}{2},\ e_1 = \omega_1 + \varepsilon,\ e_1 + e_2 = \omega_6 + 2\varepsilon,\ e_1 + e_2 + e_3 = 3\varepsilon$

Case II: The semiclassical cases: $(3 \le n)$ (see also [Kn2])

$\Sigma := \{z_1 - z_2, z_2 - z_3, \ldots, z_{n-1} - z_n, z_n\}$

$\Sigma^\vee = \{e_1, e_1 + e_2, e_1 + e_2 + e_3, \ldots, e_1 + e_2 + \ldots + e_n\}$

$\ell := z_1 + z_3 + z_5 + \ldots$

$W := \{\pi \in S_n \mid \forall i : \pi(i) - i \text{ even}\} = \langle s_{13}, s_{24}, \ldots, s_{n-2\,n}\rangle$

$\Delta^+ = \{z_i - z_j \mid 1 \le i < j \le n, i - j \text{ even}\}$

$\Phi^+ = \{z_i - z_j \mid 1 \le i < j \le n, i - j \text{ odd}\} \cup \{z_i \mid 1 \le i \le n, n - i \text{ even}\}$

$\pm W$-orbits of Σ: $[R, -R, R, -R, \ldots, S]$

$\rho = ((n-1)r + s, (n-2)r + s, \ldots, r + s, s)$

$GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$ on $(\mathbb{C}^p \otimes \mathbb{C}^q) \oplus \mathbb{C}^q$ with $1 \le p$, $2 \le q$

$p < q$: $n = 2p + 1$, $r = \frac{1}{2}$, $s = \frac{q-p}{2}$, $e_{2i} = \varepsilon_i$ $(i = 1, \ldots, p)$, $e_{2i-1} = \varepsilon'_i$ $(i = 1, \ldots, p+1)$

$p \ge q$: $n = 2q$, $r = \frac{1}{2}$, $s = \frac{p-q+1}{2}$, $e_{2i} = \varepsilon_i$ $(i = 1, \ldots, q)$, $e_{2i-1} = \varepsilon'_i$ $(i = 1, \ldots, q)$

$GL_1(\mathbb{C}) \times GL_q(\mathbb{C})$ on $(\mathbb{C} \otimes \mathbb{C}^q) \oplus (\mathbb{C}^q)^*$ with $2 \le q$

$n = 3,\ r = \frac{q-1}{2},\ s = \frac{1}{2},\ e_1 = -\varepsilon'_n,\ e_2 = \varepsilon + \varepsilon'_1 + \varepsilon'_n,\ e_3 = -\varepsilon'_1.$

$GL_p(\mathbb{C})$ on $\Lambda^2(\mathbb{C}^n) \oplus \mathbb{C}^n$ with $3 \le p$

$n = p,\ r = 1,\ s = \frac{1}{2},\ e_i = \varepsilon_i$

Case III: The quasiclassical cases: $(3 \le n)$

n **odd**

$V := \{z \in \mathbb{C}^{n+1} \mid z_n = 0\}$

$\Sigma := \{z_1 - z_2, z_2 - z_3, \ldots, z_n - z_{n+1}\}$

$\Sigma^\vee = \{e_1, e_1 + e_2, \ldots, e_1 + e_2 + \ldots + e_{n-1}, -e_{n+1}\}$

$\ell := \sum_{i=1}^{n+1} \frac{1}{2}(1 - 3(-1)^i) z_i = 2z_1 - z_2 + 2z_3 - + \ldots - z_{n+1}$

$W := \{\pi \in S_{n+1} \mid \forall i : \pi(i) - i \text{ even}, \pi(n) = n\} = \langle s_{13}, s_{24}, \ldots, s_{n-3\,n-1}, s_{n-1\,n+1}\rangle$

$\Delta^+ = \{z_i - z_j \mid 1 \le i < j \le n+1, i - j \text{ even}, j \ne n\}$

$\Phi^+ = \{z_i - z_j \mid 1 \le i < j \le n+1, i - j \text{ odd}\}$ (with $z_n = 0$)

$\pm W$-orbits of Σ: $[R, -R, R, -R, \ldots, R, S, -S]$

$\rho = ((n-2)r + s, (n-3)r + s, \ldots, r + s, s, 0, -s)$

n **even**

$\Sigma := \{z_1 - z_2, z_2 - z_3, \ldots, z_{n-1} - z_n, z_{n-1}\}$
$\Sigma^\vee = \{e_1, e_1 + e_2, \ldots, e_1 + e_2 + \ldots + e_{n-2}, -e_n, e_1 + e_2 + \ldots + e_n\}$
$\ell := \sum_{i=1}^n \frac{1}{2}(1 - 3(-1)^i) z_i = 2z_1 - z_2 + 2z_3 - + \ldots - z_n$
$W := \{\pi \in S_n \mid \forall i : \pi(i) - i \text{ even}\} = \langle s_{13}, s_{24}, \ldots, s_{n-2\,n}\rangle$
$\Delta^+ = \{z_i - z_j \mid 1 \le i < j \le n, i - j \text{ even}\}$
$\Phi^+ = \{z_i - z_j \mid 1 \le i < j \le n, i - j \text{ odd}\} \cup \{z_i \mid 1 \le i \le n, i \text{ odd}\}$
$\pm W$-orbits of Σ: $[R, -R, R, -R, \ldots, R, S]$
$\rho = ((n-2)r + s, (n-3)r + s, \ldots, r + s, s, -r + s)$

Remark: For $n = 3$ and $n = 4$, the structures in **II** and **III** are isomorphic.

$GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$ on $(\mathbb{C}^p \otimes \mathbb{C}^q) \oplus (\mathbb{C}^q)^*$ with $1 \le p$, $2 \le q$

$p < q$: $n = 2p+1$, $r = \frac{1}{2}$, $s = \frac{q-p}{2}$, $e_{2i-1} = \varepsilon_i$ $(i=1, \ldots, p)$, $e_{2i} = \varepsilon'_i$ $(i=1, \ldots, p)$, $e_{n+1} = \varepsilon'_q$
$p \ge q$: $n = 2q$, $r = \frac{1}{2}$, $s = \frac{p-q+1}{2}$, $e_{2i-1} = \varepsilon_i$ $(i = 1, \ldots, q)$, $e_{2i} = \varepsilon'_i$ $(i = 1, \ldots, q)$

$GL_p(\mathbb{C}) \times \mathbb{C}^*$ on $\Lambda^2(\mathbb{C}^p) \oplus (\mathbb{C}^p)^*$ with $3 \le p$

$n = p$, $r = 1$, $s = \frac{1}{2}$, $e_i = \varepsilon_i$ $(i = 1, \ldots, 2\lfloor \frac{n}{2} \rfloor)$, $e_{n+1} = \varepsilon_n$ if n is odd

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Case IVa:

$V := \{z \in \mathbb{C}^7 \mid z_2 + z_4 + z_6 + z_7 = 0\}$
$\Sigma := \{z_1 - z_2, z_2 - z_3, z_3 - z_4, z_4 - z_5, z_5 - z_6, z_4 + z_6\}$
$$\Sigma^\vee = \{(1,0,0,0,0,0,0), (1,1,0,0,0,0,-1), (1,1,1,0,0,0,-1), (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}), (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}), (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{3}{2})\}$$
$\ell := 2(z_1 + z_3 + z_5)$
$W := S_3(z_1, z_3, z_5) \times S_4(z_2, z_4, z_6, z_7)$
$\Delta^+ = \{z_i - z_j \mid (i,j) = (1,3), (1,5), (3,5), (2,4), (2,6), (2,7), (4,6), (4,7), (6,7)\}$
$$\Phi^+ = \{\mathrm{sign}(j-i)(z_i - z_j) \mid i = 1,3,5; j = 2,4,6,7\} \cup \{(z_i + z_j) \mid (i,j) = (2,4), (2,6), (4,6)\}$$
$\pm W$-orbits of Σ: $[R, -R, R, -R, R, \pm S]$
$\rho = (\frac{s}{2} + 4r, \frac{s}{2} + 3r, \frac{s}{2} + 2r, \frac{s}{2} + r, \frac{s}{2}, \frac{s}{2} - r, -\frac{3}{2}s - 3r)$

$Sp_{2p}(\mathbb{C}) \times GL_3(\mathbb{C})$ on $\mathbb{C}^{2p} \otimes \mathbb{C}^3$ with $3 \le p$

$r = \frac{1}{2}$, $s = p - 2$, $e_1 = \varepsilon'_1 + \varepsilon'_2$, $e_3 = \varepsilon'_1 + \varepsilon'_3$, $e_5 = \varepsilon'_2 + \varepsilon'_3$,
$e_2 - e_7 = \varepsilon_1 + \varepsilon_2$, $e_4 - e_7 = \varepsilon_1 + \varepsilon_3$, $e_6 - e_7 = \varepsilon_2 + \varepsilon_3$

Case IVb:

$$\Sigma := \{\tfrac{1}{2}(z_1 - z_2 - z_4 - z_6), z_2 - z_3, z_3 - z_4, z_4 - z_5, z_5 - z_6, z_5 + z_6\}$$
$$\Sigma^\vee = \{(2,0,0,0,0,0),(1,1,0,0,0,0),(1,1,1,0,0,0),(2,1,1,1,0,0),$$
$$(\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})\}$$
$$\ell := 2z_1$$
$$W := \mathsf{D}_3(z_2,z_4,z_6) \times \mathsf{C}_2(z_3,z_5)$$
$$\Delta^+ = \{z_i \pm z_j \mid (i,j) = (2,4),(2,6),(4,6)\} \cup \{z_3 \pm z_5, 2z_3, 2z_5\}$$
$$\Phi^+ = \{z_i \pm z_j \mid 2 \le i < j \le 6, i-j \text{ odd}\} \cup$$
$$\{\tfrac{1}{2}(z_1 \pm z_2 \pm z_4 \pm z_6) \mid 1 \text{ or } 3 \text{ minus signs}\}$$
$\pm W$-orbits of Σ: $[S, \pm R, \pm R, \pm R, \pm R, \pm R]$
$$\rho = (2s + 6r, 4r, 3r, 2r, r, 0)$$

$Sp_4(\mathbb{C}) \times GL_p(\mathbb{C})$ on $\mathbb{C}^4 \otimes \mathbb{C}^p$ with $4 \le p$

$r = \frac{1}{2}$, $s = \frac{p-3}{2}$, $e_3 = \varepsilon_1 + \varepsilon_2, e_5 = \varepsilon_1 - \varepsilon_2$

$2e_1{=}\varepsilon_1'{+}\varepsilon_2'{+}\varepsilon_3'{+}\varepsilon_4', 2e_2{=}\varepsilon_1'{+}\varepsilon_2'{-}\varepsilon_3'{-}\varepsilon_4', 2e_4{=}\varepsilon_1'{-}\varepsilon_2'{+}\varepsilon_3'{-}\varepsilon_4', 2e_6{=}{-}\varepsilon_1'{+}\varepsilon_2'{+}\varepsilon_3'{-}\varepsilon_4'$

Case IVc:

$$\Sigma := \{z_1 - z_2, z_2 - z_3, z_3 - z_4, z_4 - z_5, z_4 + z_5\}$$
$$\Sigma^\vee = \langle (1,0,0,0,0),(1,1,0,0,0),(1,1,1,0,0),(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})\rangle$$
$$\ell := 2(z_1 + z_3 + z_5)$$
$$W := S_3(z_1,z_3,z_5) \times \mathsf{C}_2(z_2,z_4)$$
$$\Delta^+ = \{z_i - z_j \mid (i,j) = (1,3),(1,5),(3,5)\} \cup \{z_2 \pm z_4, 2z_2, 2z_4\}$$
$$\Phi^+ = \{z_i \pm z_j \mid 1 \le i < j \le 5, i-j \text{ odd}\}$$
$\pm W$-orbits of Σ: $[R, -R, R, -R, R]$
$$\rho = (4r, 3r, 2r, r, 0)$$

$Sp_4(\mathbb{C}) \times GL_3(\mathbb{C})$ on $\mathbb{C}^4 \otimes \mathbb{C}^3$

$r = \frac{1}{2}$, $e_2 = \varepsilon_1 + \varepsilon_2, e_4 = \varepsilon_1 - \varepsilon_2$, $e_1 = \varepsilon_1' + \varepsilon_2', e_3 = \varepsilon_1' + \varepsilon_3', e_5 = \varepsilon_2' + \varepsilon_3'$

Case V: $(1 \le b \le a \le 3)$

$$\Sigma = \{z_1 - z_2, \ldots, z_{a-1} - z_a, z_1' - z_2', \ldots, z_{b-1}' - z_b', \\ z_a + z_b' - z'', z_a - z_b' + z'', -z_a + z_b' + z''\}$$

$$\Sigma^\vee = \{e_1, e_1 + e_2, \ldots, e_1 + e_2 + \ldots + e_{a-1}, e_1', e_1' + e_2', \ldots, e_1' + e_2' + \ldots + e_{b-1}', \\ \tfrac{1}{2}\sum e_i + \tfrac{1}{2}\sum e_i', \tfrac{1}{2}\sum e_i + \tfrac{1}{2}e'', \tfrac{1}{2}\sum e_i' + \tfrac{1}{2}e''\}$$

$$\ell := 2(z_1 + z_1')$$

$$W := (\mathbb{Z}/2\mathbb{Z})^{a+b-1} = \{(z_1, \pm z_2, \ldots, \pm z_a, z_1', \pm z_2', \ldots, \pm z_b', \pm z'')\}$$

$$\Delta^+ = \{2z_2, \ldots, 2z_a, 2z_2', \ldots, 2z_b', 2z''\}$$

$$\Phi^+ = \{z_i \pm z_{i+1} \mid 1 \le i < a\} \cup \{z_i' \pm z_{i+1}' \mid 1 \le i < b\} \cup \\ \{\pm z_a \pm z_b' \pm z'' \mid \text{at most one minus sign}\}$$

$\pm W$-orbits of Σ: $a \ge b > 1$: $[R_1, \pm R_2, \ldots, \pm R_{a-1}, R_1', \pm R_2', \ldots, \pm R_{b-1}', \pm S, \pm S, \pm S]$

$a > b = 1$: $[R_1, \pm R_2, \ldots, \pm R_{a-1}, S, -S, S]$

$$\rho = (r_1 + \ldots + r_{a-1} + s, r_2 + \ldots + r_{a-1} + s, \ldots, r_{a-1} + s, s; \\ r_1' + \ldots + r_{b-1}' + s, r_2' + \ldots + r_{b-1}' + s, \ldots, r_{b-1}' + s, s; s) \text{ (for } a > b \ge 1)$$

Remark: The cases $(a, b) = (1, 1)$ and $(2, 1)$ are isomorphic to the cases $n = 3$ and $n = 4$ of **Case II**, respectively.

$(Sp_{2p}(\mathbb{C}) \times \mathbb{C}^*) \times GL_2(\mathbb{C})$ on $(\mathbb{C}^{2p} \otimes \mathbb{C}^2) \oplus \mathbb{C}^2$ with $2 \le p$

$a = 3$, $b = 1$, $r_1 = \frac{1}{2}$, $r_2 = p - 1$, $s = \frac{1}{2}$

$e_1 = 2\varepsilon + \omega_2'$, $e_2 = \omega_2$, $e_3 = \alpha_1$, $e_1' = \omega_2'$, $e'' = \alpha_1'$

$GL_p(\mathbb{C}) \times SL_2(\mathbb{C}) \times GL_q(\mathbb{C})$ on $(\mathbb{C}^p \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^q)$ with $2 \le p, q$

$a = 2$, $b = 2$, $r_1 = \frac{p-1}{2}$, $r_1' = \frac{q-1}{2}$, $s = \frac{1}{2}$

$e_1 = \omega_2$, $e_2 = \alpha_1$, $e_1' = \omega_2''$, $e_2' = \alpha_1''$, $e'' = 2\omega'$

$(Sp_{2p}(\mathbb{C}) \times \mathbb{C}^*) \times SL_2(\mathbb{C}) \times GL_q(\mathbb{C})$ on $(\mathbb{C}^{2p} \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^q)$ with $2 \le p, q$

$a = 3$, $b = 2$, $r_1 = \frac{1}{2}$, $r_2 = p - 1$, $r_1' = \frac{q-1}{2}$, $s = \frac{1}{2}$

$e_1 = 2\varepsilon$, $e_2 = \omega_2$, $e_3 = \alpha_1$, $e_1' = \omega_2''$, $e_2' = \alpha_1''$, $e'' = 2\omega'$

$(Sp_{2p}(\mathbb{C}) \times \mathbb{C}^*) \times SL_2(\mathbb{C}) \times (Sp_{2q}(\mathbb{C}) \times \mathbb{C}^*)$ on $(\mathbb{C}^{2p} \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{2q})$ with $2 \le p, q$

$a = 3$, $b = 3$, $r_1 = \frac{1}{2}$, $r_2 = p - 1$, $r_1' = \frac{1}{2}$, $r_2' = q - 1$, $s = \frac{1}{2}$

$e_1 = 2\varepsilon$, $e_2 = \omega_2$, $e_3 = \alpha_1$, $e_1' = 2\varepsilon''$, $e_2' = \omega_2''$, $e_3' = \alpha_1''$, $e'' = 2\omega'$

$Spin_8(\mathbb{C}) \times \mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^8_+ \oplus \mathbb{C}^8_-$

$a = 2$, $b = 2$, $r_1 = \frac{1}{2}$, $r_1' = \frac{1}{2}$, $s = \frac{3}{2}$

$e_1 = 2\varepsilon$, $e_2 = \varepsilon_1 - \varepsilon_4$, $e_1' = 2\varepsilon'$, $e_2' = \varepsilon_1 + \varepsilon_4$, $e'' = \varepsilon_2 + \varepsilon_3$

Case VIa:

$\Sigma := \{z_1 - z_2, z_2 - z_3, 2z_3\}$
$\Sigma^\vee = \{(1,0,0), (1,1,0), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$
$\ell := 2z_1$
$W := (\mathbb{Z}/2\mathbb{Z})^2 = \{(z_1, z_2, z_3) \mapsto (z_1, \pm z_2, \pm z_3)\}$
$\Delta^+ = \{2z_2, 2z_3\}$
$\Phi^+ = \{z_1 \pm z_2, z_2 \pm z_3, 2z_3\}$
$\pm W$-orbits of Σ: $[R, \pm S, \pm T]$
$\rho = (r + s + \frac{t}{2}, s + \frac{t}{2}, \frac{t}{2})$

$Sp_{2p}(\mathbb{C}) \times GL_2(\mathbb{C})$ on $\mathbb{C}^{2p} \otimes \mathbb{C}^2$ with $2 \le p$
$r = \frac{1}{2}$, $s = p - 1$, $t = 1$, $e_1 = \omega_2'$, $e_2 = \omega_2$, $e_3 = \alpha_1 + \alpha_1'$

$Spin_9(\mathbb{C}) \times \mathbb{C}^*$ on $\mathbb{C}^{16}$
$r = \frac{1}{2}$, $s = 2$, $t = 3$, $e_1 = 2\varepsilon$, $e_2 = \omega_1$, $e_3 = -\omega_1 + 2\omega_4$

Case VIb:

$\Sigma := \{z_1 - z_2, z_2 - z_3, z_3 - z_4, z_3 + z_4\}$
$\Sigma^\vee = \{(1,0,0,0), (1,1,0,0), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$
$\ell := 2z_1$
$W := (\mathbb{Z}/2\mathbb{Z})^2 = \{(z_1, z_2, z_3, z_4) \mapsto (z_1, \pm z_2, \pm z_3, z_4)\}$
$\Delta^+ = \{2z_2, 2z_3\}$
$\Phi^+ = \{z_1 \pm z_2, z_2 \pm z_3, z_3 \pm z_4\}$
$\pm W$-orbits of Σ: $[R, \pm S, T, -T]$
$\rho = (r + s + t, s + t, t, 0)$

$Sp_{2p}(\mathbb{C}) \times \mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^{2p} \oplus \mathbb{C}^{2p}$ with $2 \le p$
$r = \frac{1}{2}$, $s = p - 1$, $t = \frac{1}{2}$, $e_1 = \varepsilon + \varepsilon'$, $e_2 = \omega_2$, $e_3 = \alpha_1$, $e_4 = \varepsilon - \varepsilon'$

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